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# The Strong Perfect-Graph Conjecture Is True for $K_{1,3}$-Free Graphs

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The partition number $\theta$ of a graph $G$ is the minimum number of cliques which cover the points of $G$. The independence number $\alpha$ of $G$ is the maximum number of points in an independent (stable) set of $G$. A graph $G$ is said to be perfect if $\theta(H) = \alpha(H)$ for every induced subgraph $H$ of $G$. Berge's strong perfect-graph conjecture states that $G$ is perfect iff $G$ does not contain $C_{2n+1}$ and $\bar{C}_{2n+1}$, $n \geqslant 2$ as an induced subgraph. In this paper we show that this conjecture is true for graphs which do not have $K_{1,3}$ as an induced subgraph. The line graphs thus belong to the class of graphs for which the conjecture is true.

## 1. INTRODUCTION

We consider only finite, loopless graphs without multiple edges. Thus we adopt the notation $G = (V, E)$ where $V$ is the finite vertex set of the graph $G$ and the edge set $E$ is a specified set of 2-element subsets of $V$. We denote $|V| = p$ and $|E| = q$. The complement of $G$ is the graph $\bar{G} = (V, \bar{E})$, where $\bar{E}$ is the complement of $E$ in the set of all 2-element subsets of $V$. To simplify the notation, the 2-element subset $\{u, v\}$ of $V$ is denoted by $uv$. Unless otherwise specified $V$, $E$, and $\bar{E}$ will have the above meaning with respect to the graph $G$ under discussion. When a graph $H$ has to be specified we use the more elaborate notation $V(H)$, $E(H)$ and $\bar{E}(H)$.

For any $U \subseteq V$, $\langle U \rangle$ denotes an induced subgraph of $G$ on $U$. If $H$ is a graph on fewer points than $G$, $G$ is said to be $H$-free if $G$ has no induced subgraph $H$. For $v \in V$ let $N(v) = \{u \in V \mid uv \in E\}$, $\bar{N}(v) = \{u \in V \mid uv \notin E\}$ and $d(v) = |N(v)|$. When the graph $G$ has to be specified we use the notation $N(G, v)$, $\bar{N}(G, v)$, and $d(G, v)$. For convenience of later reference

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we set $H_1(v) = \langle N(v) \rangle$, $H_2(v) = \langle \overline{N}(v) \rangle$, $H_3(v) = \langle N(\overline{G}, v) \rangle = \langle \overline{\overline{N}(v)} \rangle$, and $H_4(v) = \langle \overline{N}(\overline{G}, v) \rangle = \langle N(v) \rangle$ (see Fig. 1).

If $U = \{u_1, u_2, \ldots, u_k\} \subseteq V$ is such that no $u_i u_j \in E$, $U$ is called an *internally stable set* (*independent set*). A maximum independent set is called a MISS and the cardinality of a MISS is called *the independence number* (*stability number*) of $G$ and is denoted by $\alpha$. A *clique* is a maximal complete subgraph, and the maximum size (number of vertices) of a clique is the *density* (*or clique number*) of $G$, denoted by $\omega$. The minimum number of cliques that cover the points of $G$ is same as the minimum number of disjoint complete subgraphs that cover the points of $G$ [1]. A minimum number of cliques covering the vertices of $G$ will be called a *$\theta$-cover* of $G$. The *chromatic number* $\chi$ of $G$ is the minimum number of colors necessary to color the vertices of $G$ such that no two adjacent vertices are colored alike. A *$\chi$-coloring* of $G$ is a coloring of $G$ using minimum number of colors. A *color class in* $G$ is the set of all vertices of $G$ receiving the same color in a coloring of $G$. If $\pi$ is a $\chi$-coloring of $G$ we denote by $C_u(\pi)$ the color class of $G$ in $\pi$ containing $u$. *Unless otherwise specified* the parameters always refer to the graph $G$.

Berge [1] defined a graph to be $\alpha$-perfect if $\alpha(H) = \theta(H)$ holds for every induced subgraph $H$ of $G$ and to be $\chi$-perfect if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. A graph is said to be perfect if it is both $\alpha$-perfect and $\chi$-perfect.

In [3], Lovász settled in the affirmative the *weak perfect-graph conjecture*, that a graph is $\alpha$-perfect iff it is $\chi$-perfect. Because of this we refer to such graphs simply as perfect graphs. This result was strengthened by Lovász in [4] by proving

THEOREM 1. *For a graph the following conditions are equivalent*: *For every induced subgraph $H$ of $G$*

(i) $\chi(H)\,\alpha(H) \geqslant |V(H)|$;

(ii) $\chi(H) = \omega(H)$;

(iii) $\theta(H) = \alpha(H)$.

*The strong perfect-graph conjecture*, as it now stands, asserts that a graph is perfect if it has neither $C_{2n+1}$ nor $\overline{C}_{2n+1}$ $(n \geqslant 2)$ as an induced subgraph.

A graph $G$ is said to be *critical* if it is not perfect, and $G - v$ is perfect for every $v \in V$. An easy consequence of Lovász' result is the

COROLLARY 1. *$G$ is critical iff $\overline{G}$ is critical.*

Sachs investigated the perfect-graph conjecture at some length in [5].

Some results either explicitly obtained by him or implied by his suggestions are collected below for ready reference.

THEOREM 2. *If $G$ is critical then*

(i) $\alpha \geqslant 2$.

(ii) *For any complete subgraph $Q$ in $G$, $\alpha(G - Q) = \alpha$. In particular, $\alpha(G - v) = \alpha$ for any $v \in V$.*

(iii) *For any $v \in V$, there are at least $\alpha$ distinct* MISSes *containing $v$.*

(iv) *For any $v \in V$, $\alpha(H_2(v)) = \alpha - 1$. Each member of a $\theta$-cover for $H_2(v)$ contains at least two points.*

(v) *For any $v \in V$, $d(v) \leqslant p - 2\alpha + 1$.*

(vi) *For any $v \in V$, $H_2(v)$ and $H_4(v)$ are connected.*

Since, by Corollary 1, the complement of a critical graph is also critical we get the following translation of the above theorem to $\bar{G}$.

THEOREM 3. *If $G$ is critical then*

(i) $\omega \geqslant 2$.

(ii) *For any independent set $S$ of $G$, $\omega(G - S) = \omega$. In particular, $\omega(G - v) = \omega$, for any $v \in V$.*

(iii) *For any $v \in V$, there are at least $\omega$ maximum cliques containing $v$.*

(iv) *For any $v \in V$, $\omega(H_1(v)) = \omega - 1$.*

(v) *For any $v \in V$, $d(v) \geqslant 2\omega - 2$.*

THEOREM 4. *If $\omega = 2$ (that is, $G$ is triangle free), $G$ is perfect iff it has no induced odd cycle $C_{2n+1}$, $n \geqslant 2$.*

*Proof.* (i) $G$ has no odd induced cycle and $G$ is triangle free $\Rightarrow G$ has no odd cycle $\Rightarrow G$ is bipartite, and hence perfect.

(ii) If $G$ has an odd induced cycle $H$, $H$ is an imperfect subgraph of $G$, so that $G$ cannot be perfect.

COROLLARY 2. *If $G$ is critical and $\omega = 2$, then $G = C_{2n+1}$, $n \geqslant 2$.*

COROLLARY 3. *If $G$ is critical and $\alpha = 2$, then $G = \bar{C}_{2n+1}$, $n \geqslant 2$.*

In the next section we prove a few more preliminary results on critical graphs and a lemma which plays a key role in the proof of the main Theorem 7 of section 3, asserting the validity of the strong conjecture for $K_{1,3}$-free graphs.

## 2. Some Preliminary Results and a Basic Lemma

Theorem 5. *If $G$ is critical, then $\omega\alpha = p - 1$.*

*Proof.* Since $G$ is critical, $G - v$ is perfect for every $v \in V$. Hence by Theorem 1(i),

$$\omega(G - v)\,\alpha(G - v) \geqslant |\,V(G - v)| = p - 1.$$

But by 2(ii) and 3(ii) this becomes

$$\omega\alpha \geqslant p - 1.$$

Again by Theorem 1, $G$ is imperfect implies there is an induced subgraph $H$ of $G$ such that $\omega(H)\,\alpha(H) < |\,V(H)|$. Since for every $v \in V$, $G - v$ is perfect, $H$ cannot be a subgraph of $G - v$. Thus $H = G$ and $\omega\alpha < p$. That is $\omega\alpha \leqslant p - 1$, completing the proof.

Corollary 4. *For every $v \in V$ in a critical graph $G$, each color class in a $\chi$-coloring of $G - v$ is a* MISS.

*Proof.* $G$ is critical $\Rightarrow G - v$ is perfect $\Rightarrow \omega(G - v) = \chi(G - v)$ (Theorem 1), that is $\omega(G) = \chi(G - v)$.

But, $G$ is critical $\Rightarrow \omega\alpha = p - 1$ (Theorem 4)

$\Rightarrow \chi(G - v)\,\alpha = p - 1$ (using previous step)

$\Rightarrow \chi(G - v)\,\alpha(G - v) = p - 1$ (since $\alpha(G - v) = \alpha$)

$\Rightarrow$ Every color class in a $\chi$-coloring of $G - v$ is a MISS.

Lemma 1. *Let $G$ be a graph with the following properties*:

(i) *$G$ has a spanning cycle $v_1, v_2, \ldots, v_{2n+1}$*;

(ii) $\langle v_{2n}, v_{2n+1}, v_1, v_2, v_3, v_4\rangle = P_6$;

(iii) $\langle V(G) - \{v_1, v_2\}\rangle = P_{2n-1}$;

(iv) *$G$ is $K_{1,3}$-free.*

*Then $G$ has an odd cycle $C_{2n+1}$, $n \geqslant 2$.*

*Proof.* Let $T = \{v_{2n}, v_{2n+1}, v_1, v_2, v_3, v_4\}$, $U_1 = N(v_1) - \{v_2, v_{2n+1}\}$ and $U_2 = N(v_2) - \{v_1, v_3\}$. Then $|\,U_1| \leqslant 2$ and $|\,U_2| \leqslant 2$. For, conditions (ii) and (iii) ensure that $U_1 \subseteq V - T$ and if $\{v_r\ \ v_s, v_t\} \subseteq U_1$, then at least a pair of points, say $v_r$, $v_t$ are nonadjacent. But this implies that $\langle v_1, v_{2n+1}, v_r, v_t\rangle$ is a $K_{1,3}$, violating condition (iv). The same argument also yields the fact that $\langle U_1\rangle = K_2$ or $K_1$ or $\varnothing$, and a similar result about

$\langle U_2\rangle$. If $U_1 = \{v_r\}$, $\langle v_r, v_1, v_{r-1}, v_{r+1}\rangle = K_{1,3}$. Thus $\langle U_1\rangle \neq K_1$; and similarly $\langle U_2\rangle \neq K_1$. The remaining cases to be considered are

(i) $|U_1| = |U_2| = 0$,

(ii) $|U_1| = |U_2| = 2$,

(iii) $|U_1| = 2, |U_2| = 0$,

(iv) $|U_1| = 0, |U_2| = 2$.

In case (i), $G = C_{2n+1}$ and there is nothing to prove. In case (ii), if $U_1 = \{v_t, v_{t+1}\}$, then $v_2v_t$ and $v_2v_{t+1}$ are in $E$. If not $\langle v_1, v_2, v_{2n+1}, v_t\rangle = K_{1,3}$, etc., so that $\langle v_1, v_2, v_t, v_{t+1}\rangle$ is a $K_4$. But then either $\langle v_1, v_{2n+1}, \ldots, v_{t+1}\rangle$ or $\langle v_2, v_3, \ldots, v_t\rangle$ is an odd induced cycle, thus proving the lemma. In case (iii), if $U_1 = \{v_t, v_{t+1}\}$, $\langle v_1, v_2, v_{2n+1}, v_t\rangle = K_{1,3}$, so that this case does not arise. Similarly case (iv) is also ruled out.

## 3. The Strong Perfect-graph Conjecture for $K_{1,3}$-free Graphs

We split the long proof of the main theorem into a number of lemmas.

Lemma 2. *If $G$ is critical and $K_{1,3}$-free, then for every $v \in V$, $d(v) = 2\omega - 2$.*

*Proof.* By Theorem 3(v), $d(v) \geqslant 2\omega - 2$. Since $G$ is $K_{1,3}$-free, $\alpha(H_1(v)) \leqslant 2$. But $\alpha(H_1(v)) \neq 1$, for the contrary assumption implies $\langle\{v\} \cup N(v)\rangle$ is a clique $Q$ and $\alpha(G - Q) = \alpha - 1$, violating 2(ii). Thus $\alpha(H_1(v)) = 2$. Now $H_1(v)$ is perfect and, by 1(i), $\omega(H_1)\,\alpha(H_1) \geqslant |V(H_1)| = d(v)$. But by Theorem 3(iii), $v$ is in a maximum clique of $G$ so that $\omega(H_1) = \omega - 1$. This gives $(\omega - 1)\,2 \geqslant d(v)$ and the proof is complete.

Lemma 3. *Let $G$ be a $K_{1,3}$-free graph and $v$ be any point of $G$. Let $w$ be any arbitrary point in $N(\bar{G}, v)$ and consider any $\chi$-coloring $\pi$ of $\bar{G} - w$. Let $C_v = C_{1v} \cup \{v\}$, be the color class containing $v$ in $\pi$ and $C_{2v} = \bar{N}(\bar{G}, v) - C_v$. Then $C_{2v}$ is an independent set of cardinality $\omega - 1$.*

*Proof.* The assertion of the lemma will be proved if we show that $\langle C_{2v}\rangle$ is a complete subgraph in $G$ of cardinality $\omega - 1$.

From Lemma 2, $|C_{1v} \cup C_{2v}| = 2\omega - 2$. Also by corollary $|C_{1v} \cup \{v\}| = \omega$, so that $|C_{1v}| = \omega - 1$. Thus $|C_{2v}| = \omega - 1$, and it only remains to be verified that $\langle C_{2v}\rangle$ is a complete subgraph in $G$.

Suppose the contrary, i.e., there are points $r, s \in C_{2v}$ such that $rs \notin E$. Consider any $\chi$-coloring $\Sigma$ of $G - w$ where $w$ is the point of the lemma considered as the point of $G$. This induces a coloring of $H_1(v)$. But, as already seen, $\alpha(H_1(v)) = 2$. Since $H_1(v)$ is perfect, $\theta(H_1(v)) = 2$; i.e., there exist two disjoint $K_{\omega-1}$'s covering the vertices of $H_1(v)$. Therefore, in any minimal coloring of $H_1(v)$, each color is assigned to two points. Since $\langle C_{1v}\rangle$ is a clique in $H_1(v)$, each point of $C_{1v}$ recieves one color, the other point receiving the same color in $H_1(v)$ being in $C_{2v}$. Thus no two points of $C_{2v}$ receive the same color in any coloring of $H_1(v)$. Applying this to the coloring of $H_1(v)$ induced by the $\chi$-coloring $\Sigma$ of $G - w$, we suppose that $C_r$ and $C_s$ are the color classes of $\Sigma$ in $G - w$ containing $r$ and $s$, respectively. By Corollary 4, $| C_r | = | C_s | = \alpha$. Let $r'$ and $s'$ be the points of $C_{1v}$ in the color classes $C_r$ and $C_s$. Then all other points of $C_r$ and $C_s$ are in $H_{2(v)}$ (see Fig. 1).

Now $rs'$ and $sr'$ are in $E$. For, if not, $\langle v, r, s, s'\rangle = K_{1,3}$ and $\langle v, r', r, s\rangle = K_{1,3}$. Further, there is an $r'' \in C_r - \{r, r'\}$ such that $sr'' \in E$. For, otherwise $\{s\} \cup (C_r - \{r\})$ is an independent set in $G - w - C_v$ of cardinality $\alpha$, violating the fact that $\alpha(G - w - C_v) = \alpha(G - w) - 1 = \alpha - 1$. Similarly, there is an $s'' \in C_s - \{s, s'\}$ such that $rs'' \in E$. Since $r''$ and $s''$ are in $H_2(v)$, which is connected (by Theorem 2(vi)), there is a shortest path $P''$ between $r''$ and $s''$ in $H_2(v)$.

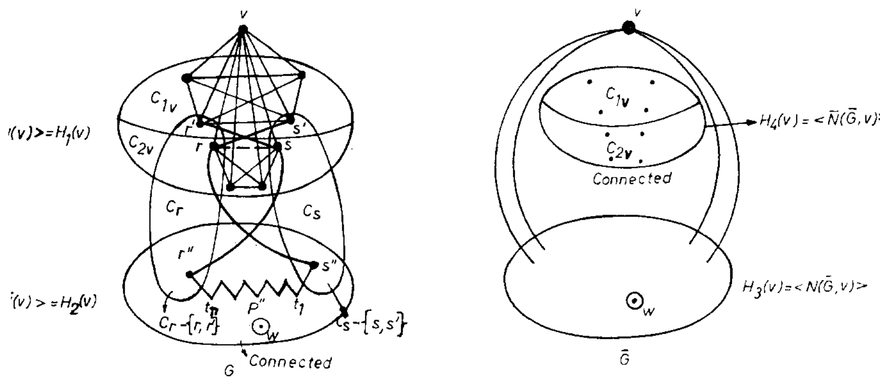


FIGURE 1

Let $P'' = s''t_1t_2 \cdots t_nr''$ with $T = \{t_1, t_2, \ldots, t_n\}$. Then it is easily seen that $s''r' \notin E$ and $r''s' \notin E$, for the contrary assumptions lead to the conclusions $\langle s'', r, r', r''\rangle = K_{1,3}$ and $\langle r'', s, s', s''\rangle = K_{1,3}$.

Now we consider different possibilities concerning $T$ and show that each can lead to a contradiction, thus discrediting the hypothesis that $rs \notin E$ and establishing the lemma.

*Case* (i). $T = \varnothing$. This implies that $\langle s'', r, v, s, r''\rangle = C_5$, contradicting the fact that a critical graph cannot have $C_{2n+1}$, $n \geqslant 2$ as a proper induced subgraph.

Therefore $T \neq \varnothing$, and let $T' = \{t_i \in T \mid t_i r \in E \text{ or } t_i s \in E\}$.

*Case* (ii). $T' = \varnothing$. Two subcases now arise. If $|T|$ is even, then $\langle T \cup \{s'', r, v, s, r''\}\rangle$ is an odd induced cycle. If $|T|$ is odd, then $\langle T \cup \{s'', r, s', r', s, r''\}\rangle$ satisfies the conditions of Lemma 1, and hence contains an odd induced cycle $C_{2m+1}$, $m \geqslant 2$.

Therefore we assume $T' \neq \varnothing$. Now it can easily be observed that $r$ can be adjacent only to $t_1 \in T$. For if $rt_i \in E$ for $t_i \in T - \{t_1\}$, then $\langle r, s'', t_i, v\rangle = K_{1,3}$. Similarly $s$ can be adjacent only to $t_n$ in $T$.

*Case* (iii). There exists a $t \in T'$ such that $tr \in E$ and $ts \in E$. This implies that $t = t_1 = t_n$. But then $tr' \notin E$ and $ts' \notin E$ (otherwise $\langle t, r, r', r''\rangle = K_{1,3}$ and $\langle t, s, s', s''\rangle = K_{1,3}$). Thus $\langle t, r, s', r', s\rangle = C_5$.

*Case* (iv). $T'$ consists only of points $t \in T$ such that either $tr \in E$ and $ts \notin E$ or $ts \in E$ and $tr \notin E$.

Let $P$ be the shortest $r - s$ path using the points of $P''$ and let $P' = P - \{r, s\}$. If $|P'|$ is even, then $\langle \{v\} \cup P\rangle$ is an induced odd cycle $C_{2m+1}$, $m \geqslant 2$. Thus we assume $|P'|$ is odd.

Now $P'$ is $P''$, or $P'' - r''$, or $P'' - s''$, or $P'' - s'' - r'' = T$. As the subcase $P' = P''$ is case (ii), which is ruled out, we consider only the latter three cases and show in each case that $\langle t_j, r, s', r', s, t_k\rangle = P_6$ where $t_j$ and $t_k$ are the first and the last points of $P'$.

*Subcase* (iv)(a). $P' = P'' - r''$ (see Fig. 2(a)). Then $T \cup \{s''\} = P'$ so that $|T|$ is even. Thus $|T| \geqslant 2$. Now $\langle r, s', r', s\rangle = P_4$, $s''r' \notin E$, $s''s \notin E$ and $s''t_n \notin E$. Further, $rt_n \notin E$ (for case (iii) has been ruled out). Also $r't_n \notin E$; for

$$\begin{aligned} r't_n \in E &\Rightarrow r't_{n-1} \in E \text{ (otherwise } \langle t_n, r', r'', t_{n-1}\rangle = K_{1,3}) \\ &\Rightarrow \text{also } r't \notin E \text{ for any } t \in P' - \{t_n, t_{n-1}\} \\ &\quad \text{(otherwise } \langle r', v, t_n, t\rangle = K_{1,3}) \\ &\Rightarrow \langle P' \cup \{r, v, r'\} - \{t_n\}\rangle = C_{2m+1}, m \geqslant 2. \end{aligned}$$

Again $s't_n \notin E$, for, otherwise, $\langle s', r, t_n, r''\rangle = K_{1,3}$. Thus we have $\langle s'', r, s', r', s, t_n\rangle = P_6$.

*Subcase* (iv)(b). $P' = P'' - s''$. This is similar to the previous one.

*Subcase* (iv)(c). $P' = P'' - \{r'', s''\} = T$ (see Fig. 2(b)). It is easily seen that $t_j = t_1$ and $t_k = t_n$ and, as in subcase (iv)(a), we prove that $\langle t_1, r, s', r', s, t_n\rangle = P_6$.

Now let $H = \langle P' \cup \{r, s', r', s\}\rangle$. Then $H$ contains a spanning odd cycle $C_{2m+1}$ and $\langle V(H) - \{r', s'\}\rangle$ is a $P_{2m-1}$ and $H$ does not have $K_{1,3}$ as an induced subgraph. Thus $H$ satisfies all the conditions of Lemma 1 and therefore has an induced odd cycle $C_{2k+1}$, $k \geqslant 2$, which is also an induced odd cycle of $G$, violating criticality of $G$.

Thus each case leads to a contradiction and the lemma is established.

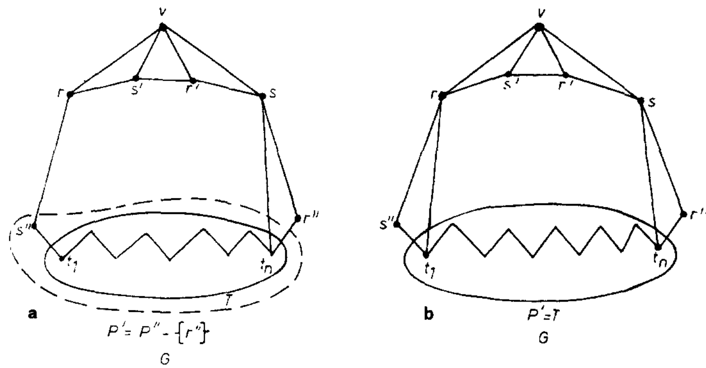


FIGURE 2

COROLLARY 5. *With the assumptions and notations in Lemma* 3, *the color class containing* $v$ *in the* $\chi$*-coloring* $\pi$ *of* $\bar{G} - w$ *is either* $C_{1v} \cup \{v\}$ *or* $C_{2v} \cup \{v\}$.

The proof follows by Lemma 3 and the fact that a connected bipartite graph has a unique bipartition.

LEMMA 4. *With the assumptions and notations in Lemma* 3, $C_{2v}$ *is contained in some color class in the* $\chi$*-coloring* $\pi$ *of* $\bar{G} - w$.

*Proof.* The lemma asserts that all the points of $C_{2v}$ are colored alike in $\pi$. If not, the points of $C_{2v}$ may be partitioned as $C_{2v} = L_1 \cup L_2 \cup \cdots \cup L_k$ ($k \geqslant 2$) such that each $L_i$ is in a different color class in $\pi$.

*Case* (i). Each $L_i$ has just one point. Let $L_i = \{u\}$, and let $C_u$ be the color class of $\pi$ containing $u$. Since $v$ and each point of $C_{2v} - \{u\}$ are assigned colors different from that of $u$, $C_u - \{u\} \subset N(\bar{G}, v)$ and $(C_{2v} - \{u\}) \cap C_u = \varnothing$. Thus $(C_u - \{u\}) \cup (C_{2v} - \{u\}) \cup \{v\} \subseteq \bar{N}(\bar{G}, u)$ and the former set has cardinality $2\omega - 2$. But $|\bar{N}(\bar{G}, u)| = |N(G, u)| = 2\omega - 2$ by Lemma 2. Thus $\bar{N}(\bar{G}, u) = (C_u - \{u\}) \cup (C_{2v} - \{u\}) \cup \{v\}$, so that $C_{1v} \subset N(\bar{G}, u)$. Thus $u$ is adjacent in $\bar{G}$ to every point of $C_{1v}$. Since $u$ is arbitrary, every point of $C_{2v}$ is adjacent to every point of $C_{1v}$; that is, $\langle C_{1v} \cup C_{2v} \rangle = K_{\omega-1,\omega-1}$ in $\bar{G}$. But this implies that $v$ belongs to only two MISSes in $\bar{G}$ (namely, $\{v\} \cup C_{1v}$ and $\{v\} \cup C_{2v}$), that is, $\omega = 2$ (by Theorem 2(iii) applied to $\bar{G}$). By Corollary 2 this implies $G = C_{2n+1}$, $n \geqslant 2$, for which the lemma is trivially true.

Incidentally, $w \in N(\bar{G}, u)$ so that $C_u - \{u\} = C_{1u}$ and $(C_{2v} - \{u\}) \cup \{v\} = C_{2u}$ (see Fig. 3a).

*Case* (ii). There is at least one $L_i$ such that $|L_i| \geqslant 2$.

We first show that each $L_i$ contains one $u_i$ such that $u_i w \in \bar{E}$. Suppose that there is an $L_i$ such that, for every $u \in L_i$, $uw \notin \bar{E}$. That is, $uw \in E$. Then $wx \in E$, for every $x \in C_u - L_i$, since otherwise $\langle u, v, w, x\rangle = K_{1,3}$ in $G$. Thus we observe that $\langle C_u \cup \{u\}\rangle$ is a complete subgraph of $G$ of cardinality $\omega + 1$, which is a contradiction.

Let $|L_k| \geqslant 2$, and let $r \in L_1$ and $s \in L_k$, such that $rw \in \bar{E}$ and $sw \in \bar{E}$, and let $C_r$ and $C_s$ be the color classes containing $r$ and $s$ respectively, in this $\chi$-coloring $\pi$ of $\bar{G} - w$. Then, in the notation earlier employed for $v$ (which can be adopted here since $rw \in \bar{E}$ and $sw \in \bar{E}$), $L_1 \subset C_r$, $L_k \subset C_s$, $C_r - L_1 \subset N(\bar{G}, v)$, and $C_s - L_k \subset N(\bar{G}, v)$. Further, $v \in C_{2r}$ and $v \in C_{2s}$, and if $L_1' = C_{2r} - (C_{2v} - L_1) - \{v\}$, and $L_k' = C_{2s} - (C_{2v} - L_k) - \{v\}$, then $L_1' \subset C_{1v}$ and $L_k' \subset C_{1v}$, so that $L_1' = C_{2r} \cap C_{1v}$ and $L_k' = C_{2s} \cap C_{1v}$ (see Fig. 3(b)). But then it is easy to calculate that $|L_1'| = |L_1| - 1$, and $|L_k'| = |L_k| - 1$, and $|L_k'| \geqslant 1$. Further, since $L_1$ and $L_k'$ are included in $C_{2s}$, $L_1 \cup L_k'$ is an independent set in $\bar{G}$. Since $r \in L_1$, this implies that $r$ is not adjacent in $\bar{G}$ to any point of $L_k'$, that is, $L_k' \subset \bar{N}(\bar{G}, r) = C_{1r} \cup C_{2r}$. But $L_k' \cap C_{1r} = \varnothing$ so that $L_k' \subset C_{2r}$. Therefore $L_k' \subset C_{2r} \cap C_{1v} = L_1'$. Let $u \in L_k' \cap L_1'$. Then $u$ is not adjacent in $\bar{G}$ to $v$ and to any point of $C_{2v} - L_1$ and to any point of $C_{2v} - L_k$, so that $\{u\} \cup \{v\} \cup C_{2v}$ is an independent set in $\bar{G}$ of cardinality $\omega + 1$, which is a contradiction.

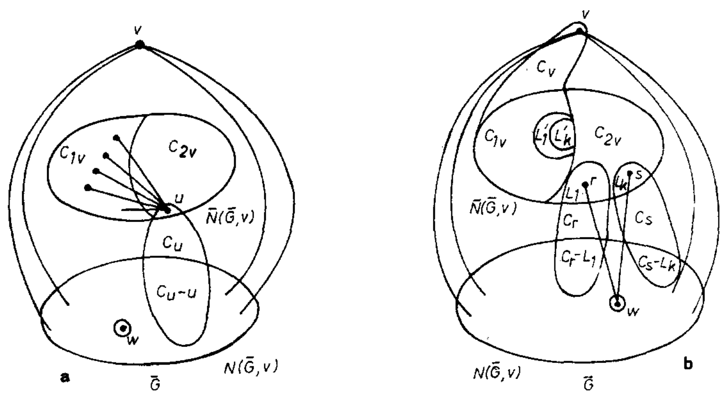


FIGURE 3

LEMMA 5. *With the assumptions and notations of Lemmas* 3 *and* 4, *let* $L = \{u \in C_{2v} \mid ux \in \bar{E}$, *for some* $x \in N(\bar{G}, v)\}$. *Then every point of* $L$ *is a pendant vertex in* $H_4(v)$.

*Proof.* By Lemma 4, $L \neq \varnothing$ and $C_{2v}$ is contained in a color class of $\pi$, which is clearly $C_{2v} \cup \{w'\}$ for some $w' \in N(\bar{G}, v)$. We now claim that any point of $L$ is adjacent in $\bar{G}$ to every point of $N(\bar{G}, v)$ except $w'$.

First we prove this for any $u \in L$ such that $uw \in \bar{E}$ (In the course of the proof of Lemma 4 we have established the existence of such a $u$). Clearly, $C_u(\pi) = C_{2v} \cup \{w'\}$ and $C_{1u}(\pi) = C_u - \{u\}$, so that $v \in C_{2u}$, and since $C_{2u}$ is an independent set of cardinality $\omega - 1$, it consists of $v$ and $\omega - 2$ points of $C_{1v}$. Thus $u$ is adjacent in $\bar{G}$ to at most one point of $C_{1v}$. But, by the connectedness of $H_4(v)$, $u$ is adjacent to at least one point of $C_{1v}$. Thus $u$ is adjacent to exactly one point of $C_{1v}$, so that $u$ is a pendant vertex in $H_4(v)$ and is adjacent to every point of $N(\bar{G}, v) - \{w'\}$.

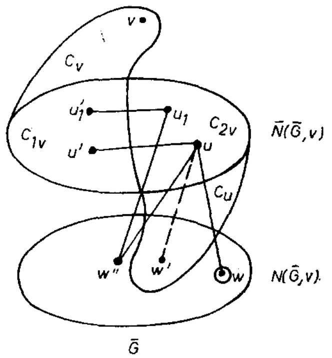


FIGURE 4

If, now, $u_1 \in L$ is such that $u_1 w'' \in \bar{E}$ for $w'' \in N(\bar{G}, v)$ but $w'' \neq w$, consider a $\chi$-coloring $\pi''$ of $G - w''$ (see Fig. 4). Since, by the previous paragraph, $u_1 w'' \in \bar{E}$, $C_u(\pi'')$ is either $C_{1u} \cup \{u\} = C_{2v} \cup \{w'\}$, or $C_{2u} \cup \{u\} = (C_{1v} - \{u'\}) \cup \{u, v\}$ (by Corollary 5, applied to $u$). But the latter choice contradicts the only two possible choices, namely $C_{1v} \cup \{v\}$ and $C_{2v} \cup \{v\}$, for $C_v(\pi'')$. Thus, $C_u(\pi'') = C_{2v} \cup \{w'\}$. But $C_{u_1}(\pi'') = C_u(\pi'')$, so that $C_{2u_1}(\pi'') = \{v\} \cup S$ where $S \subseteq C_{1v}$ with $|S| = \omega - 2$. Thus, as for $u$ earlier, with the connectedness of $H_4(v)$, we get that $u_1$ is adjacent to exactly one vertex of $C_{1v}$ and is thus a pendant vertex of $H_4(v)$. (Incidentally, $ux \in \bar{E}$, $\forall x \in N(\bar{G}, v) - \{w'\}$.)

We are now ready to prove

THEOREM 6. *If $G$ is a $K_{1,3}$-free critical graph, then $G$ is $C_{2n+1}$ or $\bar{C}_{2n+1}$, $n \geqslant 2$.*

*Proof.* As seen earlier, in the notation of Lemma 5, $L \neq \varnothing$.

Suppose $L = C_{2v}$. Then we prove that $|L| = |C_{2v}| = 1$. If $|L| \geqslant 2$, there are at least two points in $C_{2v}$, and therefore at least two points, say $r$ and $s$, in $C_{1v}$. Since every point of $C_{2v}$ is a pendant vertex (Lemma 5), there is no path between $r$ and $s$ in $H_4(v)$, violating the connectedness of $H_4(v)$. But $|C_{2v}| = 1 \Rightarrow \omega = 2 \Rightarrow G = C_{2n+1}$, $n \geqslant 2$ (by Corollary 2).

If $L \neq C_{2v}$, then there is a point $u \in C_{2v}$ not adjacent to any point of $H_3(v)$. Then, the number of points nonadjacent to $u$ in $\bar{G}$, say $\bar{d}(\bar{G}, u) \geqslant |C_{2v} - \{u\}| + |N(\bar{G}, v)| + |\{u\}|$. That is, $\bar{d}(\bar{G}, u) \geqslant \omega - 2 + (p - 2\omega + 1) + 1 = p - \omega$. But, by Lemma 2, $d(G, u) = 2\omega - 2$, so that $\bar{d}(\bar{G}, u) = 2\omega - 2$. Therefore $2\omega - 2 \geqslant p - \omega$, i.e., $p - 1 \leqslant 3\omega - 3$. Using Theorem 5, this gives $\omega\alpha \leqslant 3\omega - 3$ or $\alpha \leqslant [3 - (3/\omega)]$, since $\omega > 0$ and $\alpha$ is an integer. Since $\alpha = 1$, this implies that $\omega = 2$, that is, $G = \bar{C}_{2n+1}$, $n \geqslant 2$ (from Corollary 3). This completes the proof of the theorem.

COROLLARY 6. *If $G$ is critical and $K_{1,3}$-free, then $\bar{G}$ is also critical and $K_{1,3}$-free.*

THEOREM 7 (*Berge's strong conjecture for $K_{1,3}$-free graphs*). *If $G$ does not have either $K_{1,3}$ or $C_{2n+1}$ or $\bar{C}_{2n+1}$, ($n \geqslant 2$) as an induced subgraph, then $G$ is perfect.*

*Proof.* Suppose $G$ is imperfect. Let $H$ be an induced subgraph of $G$ with the minimum number of points for which $\theta(H) > \alpha(H)$. Then $H$ is critical and does not have $K_{1,3}$ as an induced subgraph. Then, by Theorem 6, $H = C_{2n+1}$ or $\bar{C}_{2n+1}$, $n \geqslant 2$, contradicting the hypothesis. Therefore $G$ is perfect.

COROLLARY 7. *To prove the strong perfect-graph conjecture it is enough to show that a critical graph does not have $K_{1,3}$ as an induced subgraph.*

COROLLARY 8. *A line graph $L(G)$ is perfect iff it has no $C_{2n+1}$, $n \geqslant 2$, as an induced subgraph.*

*Proof.* $L(G)$ does not have $K_{1,3}$ as an induced subgraph. Hence by Theorem 7, it is enough to prove that $L(G)$ has no $\bar{C}_{2n+1}$ as an induced subgraph (with $n \geqslant 3$; for $n = 2$, $\bar{C}_{2n+1} = C_{2n+1}$). Suppose $L(G)$ has a $\bar{C}_{2n+1}$, $n \geqslant 3$, as an induced subgraph with $C_{2n+1} = v_1 v_2 \cdots v_{2n+1}$. Then the induced subgraph $\langle v_1, v_2, v_4, v_5, v_{2n+1}\rangle$ is the graph of Fig. 5, which is a prohibited subgraph for a line graph.

COROLLARY 9. *A line graph $L(G)$ is perfect iff $G$ has no $C_{2n+1}$, $n \geqslant 2$, as a subgraph.*

*Proof.* Follows from the fact that $G$ has a subgraph $C_{2n+1}$ ($n \geqslant 2$) iff $L(G)$ has an induced subgraph $C_{2n+1}$, $n \geqslant 2$.

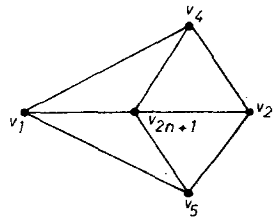


FIGURE 5

## ACKNOWLEDGMENTS

The authors are very grateful to the referees for their valuable comments on the original draft, which led to this much improved version. In particular, the proof of Lemma 3 has been greatly simplified and clarified in the revision.

Our attention has been drawn to [2, 6] and to the following

THEOREM (Las Vergnas, Sumner). *Every connected $K_{1,3}$-free graph with even number of vertices has a perfect matching.*

Perhaps this can be utilized to simplify the long proofs of some of the lemmas in this paper.

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